Indicators and Estimators in P-Adaptive Boundary Elements

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1. Introductory Remarks

When dealing with a numerical method of the finite element (F.E.) or boundary element (B.E.) type it is important to be sure about the quality of the results. One typical test is to enrich the aproximating - basis in order to see the convergence of the results. In general it is possible to proceed along two estrategies, either increasing the number of elements in the discretizing mesh or increasing the degree of - the polynomial aproximation inside the initial mesh. The first approach implies the change of the initial mesh, reducing its typical size h and is, consequently, called an h-method. The second possibility manteins the size of the elements but increases the degree p of the interpolating polinomia and this is why it is called a p-method. The p-method has been widely used in F.E. as shown by references [5, 6, 7, 8, 9, 10, 11, 14, 16], til the point of dedicating an Specialty Conference to it [7]. The possibility of h-refinement in B.E. was indicated for instance in [2] and, recently, RANK [12] presented a version of an h-adaptive - method. The p-adaptive B.E.M. has been presented in [1, 3, 4, 13].

The are several features that characterize an adaptive method. First a hierarchy of interpolating functions, then a criterion or indicator - which points-out where to refine and finally an estimator capable of - predict the degree of accuracy obtained with the computed solution. The incorporation of this philosophy to B.E.M. is only possible if changing the general isoparametric approach typical of the standar direct B.E.M. That means, for instance, that the interpolation of the geometry is, - completely independent of that of the variables. As shown in fig. 1, - where the general procedure is schematised, that step is done by a - preporcessor in an interactive way. Once the body has been defined the elements are identified. Due to the general smoothness of the solution on the boundary it is generally possible to use elements larger than those of the clasical approach; the only limitations are singularities or corners. The idea is to combine the advantages of the boundary discretization and theuse of a hierarchy of functions supported by those "macroelements".

In this respect it is important to recall that the weighting functions in B.E.M. are globally based, due to the properties of the fundamental solution, so that the aforementioned approach seems to be more natural than the previous isoparametric B.E.M.

There are several possibilities to choose hierarchies. It is possible to work with Fourier series but in general it is prefered to use polinomia. A very popular series for monodimensional problems was proposed by Peano [10] and it is formed by the linear interpolation:

$$N_o = \frac{1}{2}(1+\xi);\ N_1 = \frac{1}{2}(1-\xi) \qquad 0\leq\xi\leq 1 \qquad \ldots(1)$$

plus the family:

$$N_p = \frac{1}{p!}(\xi^p - b);\ p>2;\ \begin{matrix} b=1 & p \text{ odd} \\ b=\xi & p \text{ even} \end{matrix} \qquad \ldots(2)$$

There is some controversy about the use of that family and other authors prefer to recommend for instance:

$$N_p = \frac{1}{(p-1)!}\,\frac{1}{2^{p-2}}\,\frac{d^{p-2}}{d\xi^{p-2}}\left[(1-\xi^2)^{p-1}\right] \qquad \ldots(3)$$

For low order interpolation both approaches produce the same results and the difference is only noticeable when treating high gradients. As an exemple of the results that can be obtained, figure 2 presents the - different degrees of interpolation for a typical problem: the flow - around a cylinder of a perfect fluid constrained between parallel plates. Figure 2b shows the solution for linear fuctions on the boundary and 2c and 2d the subsequent parabolic and cubic refinements. It is - worth noticing the filter capability of the representation formula - that produces good results inside the domain even when the aproximation on the boundary is a rough one. The results of figure 2 have been obtained with a computer program prepared for a typical microcomputer and,

as shown in figure 1.
The process is organized around four different blocks. The preprocessor adjusts the geometry and the boundary conditions and the postprocessor produces plots of the partial or total solution and generates the output files. The primary solver is simply a program capable of solving the linear interpolation of the unknowns on a general boundary geometry. This block produces the first solution, the first matrix and load vector and the first estimate of the quality of the solution. After that it is possible either to analyze graphically the output or to proceed along with the adaptive block. That one is organized around the computation of the indicators for every element i.e.: a criterion is included in order to decide what elements are more critical in relation to the accuracy. Once the elements to be refined are chosen the new matrix is assembled by completing the old one along the following scheme:

$$\underset{\sim}{A}^{*}\ \underset{\sim}{\phi} = \underset{\sim}{B}^{*}\ \underset{\sim}{q} \qquad \ldots(4)$$

where:

$$\begin{array}{ccc} & & \text{Collocation points} \\ \underset{\sim}{A}^{*}\ (\underset{\sim}{B}^{*}) = & \left[\begin{array}{c|c} \begin{array}{c}\underset{\sim}{A}\\ (\underset{\sim}{B})\end{array} & \begin{array}{c}\text{New } \underset{\sim}{N} \text{ from}\\ \text{old } \underset{\sim}{x}\end{array} \\ \hline \begin{array}{c}\text{Old } \underset{\sim}{N} \text{ from}\\ \text{new } \underset{\sim}{x}\end{array} & \begin{array}{c}\text{New } \underset{\sim}{N}\\ \text{from new } \underset{\sim}{x}\end{array} \end{array}\right] & \begin{array}{c}\text{OLD}\\ \\ \text{NEW}\end{array} \\ & \begin{array}{cc}\text{OLD} & \text{NEW}\end{array} & \text{Interpolating functions} \end{array} \qquad \ldots(5)$$

As can be seen the progression of refinement produces the nesting of - the influence matrices so that all previous work can be used again. After the new system has been solved the estimator computes a new degree of accuracy and either the user or an automatic rule proceeds along a - new refinement or stops the process.

2. The indicator

As can be seen from the previous description one of the critical points in the method is the selection of a suitable indicator. In general terms the most direct approach is that proposed by Peano [10] interpreting the refinement process as an iteration one where every added degree of - freedom is simply an improvement of the solution. The idea is to start with a first initial solution (the linear one) $\underset{\sim}{\delta}_i$.

$$\underset{\sim}{K}_{ii}\ \underset{\sim}{\delta}_i = \underset{\sim}{P}_i \qquad \ldots(6)$$

after what the improved system will take the aspect:

$$\begin{bmatrix} \underset{\sim}{K}_{ii} & \underset{\sim}{K}_{ij} \\ \underset{\sim}{K}_{hi} & \underset{\sim}{K}_{hh} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\delta}_i \\ \underset{\sim}{\delta}_h \end{bmatrix} = \begin{bmatrix} \underset{\sim}{P}_i \\ \underset{\sim}{P}_h \end{bmatrix} \qquad \ldots(7)$$

nesting the previous equations.
From the second set, if only the terms in the diagonal are considered in $\underset{\sim}{K}_{hh}$, it is possible to estimate a certain $\delta_j \varepsilon \underset{\sim}{\delta}_h$ as:

$$\delta_j = \frac{P_j - \underset{\sim}{K}_{ji}\ \underset{\sim}{\delta}_i}{K_{jj}} \qquad \ldots(8)$$

The criterion described by Peano is based on the value:

$$Q_j = \frac{\sum_{i \neq j} K_{ji}\ \delta_i - P_j}{\sqrt{K_{jj}}} \qquad \ldots(9)$$

where δ_i is assumed to be equal to that of the previous step. As shown by Gago [8], Q_j can be related to the refinement in energy while working with F.E.M. The previous indicator has been incorporated to the program but substituting $\underset{\sim}{K}_{ih}^T$ in place of $\underset{\sim}{K}_{hi}$ because $\underset{\sim}{K}_{ih}$ is easily computed while forming $\underset{\sim}{K}_{ii}$. Another possibility that has been analyzed is to compare the importance of the proposed refinement in relation to the previous solution. That is, if the previous variable is called u_1 and the improved one u_2 we have:

$$u_2 - u_1 = \delta_h\ N_h \ ; \qquad u_1 = \sum_0^{h-1} \delta_k\ N_k \qquad \ldots(10)$$

so that a measure of the above mentioned importance is indicated by:

$$\varepsilon = (\int (u_2-u_1)^2 / \int u_1^{\,2})^{1/2} = \delta_h\ (\int N_h^{\,2} / \int [\sum_0^{h-1} \delta_k\ N_k]^2)^{1/2} \qquad \ldots(11)$$

Of course, to aply this criterion it is neccessary to estimate δ_h in a similar manner to that cited above. In general both indicators show -

similar trends although the second one seems to produce more reasonable guidelines.

Finally other possibility is to use the graphics capability of interactive processes along the repetitive use of the representation formula to compute selected values in areas of the boundary of special interest and to compare with the interpolated solution letting in the hands of the user the possibility of refining after inspection on the agreement of the check.

As an exemple figures 3, 4 and 5 present several refinements of a classical problem that was analyzed in [2] using progressive h refinements. The most important piece of the boundary is the macroelement AB, where a singularity appears in the normal flux distorting the results around point B.

In figure 3 the sequence shows how the singularity is progresively catched by increasing the number of hierarchic functions. In this case the family chosen was Peano's while in figure 4 the series defined by equation 3 was used. Finally figure 5 shows the results of a direct checking of the plotted results. Table 1 collects the values taken by the indicators problem of figure 2 for the confirming the above mentioned conclusions.

3. The estimator

It is clear that the value of the indicator provides a guide to stop the process, and that is the viewpoint defended by several authors. - Nontheless the success of the a-posteriori estimates developed by Babuska et al [6] for the F.E.M. are an atractive path to follow. Unfortunatly those mathematical developments have not been developed yet for the B.E.M. and our approach is being based currently on heuristic bases.

One of the procedures we have been testing is the fullfillement of an "equilibrium-type" condition i.e.: the error on the condition:

$$\int_{\partial\Omega} q = 0 \qquad \ldots(12)$$

that has to be accomplished by Dirichlet or mixed problems. Table II shows the evolution of that condition in the problem of figures 4, 5 and 6.

As can be seen the first few refinements shows a good progression that stops for higher refinements, indicating may be, that there is a saturation of the unit of measure taken. That is not strange in this problem where the difficulty is localized at a point while the measure is a global one.

Other useful possibility in laplacian problems is to measure the energy on the boundary by using the identity:

$$\int_{\Omega} \nabla\phi.\nabla\phi = -\int_{\Omega}\phi\nabla^2\phi + \int_{\partial\Omega} q\ \phi = \int_{\partial\Omega} q\ \phi \qquad \text{...(13)}$$

It has been shown elsewhere that this approach produces in occasions better results than the previous one, although aparently the situation is very problem dependent.

4. Conclusions

The paper presents the possibility of implementing a p-adaptive process with the B.E.M. Although the exemples show that good results can be obtained with a limited amount of storage and with the simple ideas explained above, more research is needed in order to improve the two main problems of the method, i.e.: the criteria of where to refine and until - what degree. Mathematically based reasoning is still lacking and will be useful to simplify the decission making. Nevertheless the method - seems promising and, we hope, opens a path for a series of research - lines of maximum interest.

Although the paper has dealt only with plane potential problem the extension to plane elasticity as well as to 3-D potential problem is - straight-forward.

5. References

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TABLE I

EVOLUTION OF THE INDICATORS FOR PROBLEM OF FIGURE 2.

ITERATION NUMBER 1 (LINEAR TO PARABOLIC)

Element number	Criterion of formulae 9	Criterion of formulae 11
1	0.1738	0.1386
2	1.699	1.257
3	1.569	1.907
4	2.643	2.109
5	1.763	1.406

ITERATION NUMBER 2 (PARABOLIC TO CUBIC)

Element number	Criterion of formulae 9	Criterion of formulae 11
1	0.529	1.195
2	4.295	9.672
3	12.31	59.81
4	5.023	11.34
5	3.894	8.78

TABLE II

EVOLUTION OF THE ESTIMATOR $\int_{\partial\Omega} q = 0$

DEGREE OF INTERPOLATION	FIG. 3 HIERARCHY EQ. 2	FIG. 4 HIERARCHY EQ. 3
LINEAR	28.6	28.6
PARABOLIC	4.2	4.2
CUBIC	3.6	3.6
QUARTIC	1.1	1.0
QUINTIC	3.8	3.75

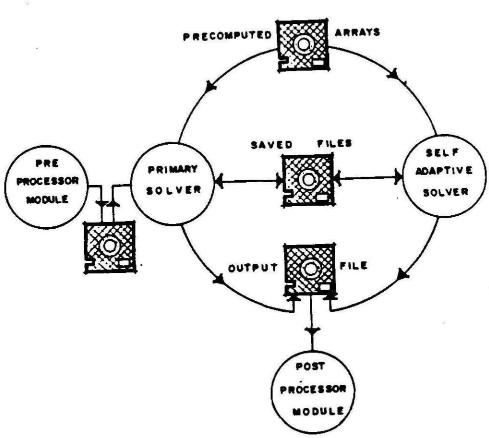

Fig. 1

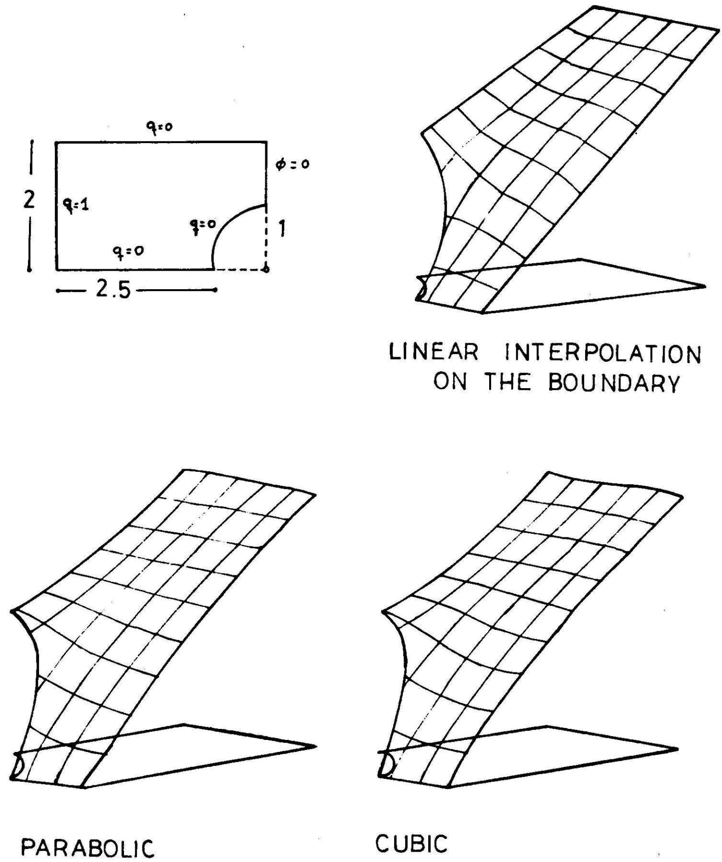
q=0
φ=0
2
q=1
q=0
q=0
1
2.5
LINEAR INTERPOLATION ON THE BOUNDARY
PARABOLIC
CUBIC

Fig. 2

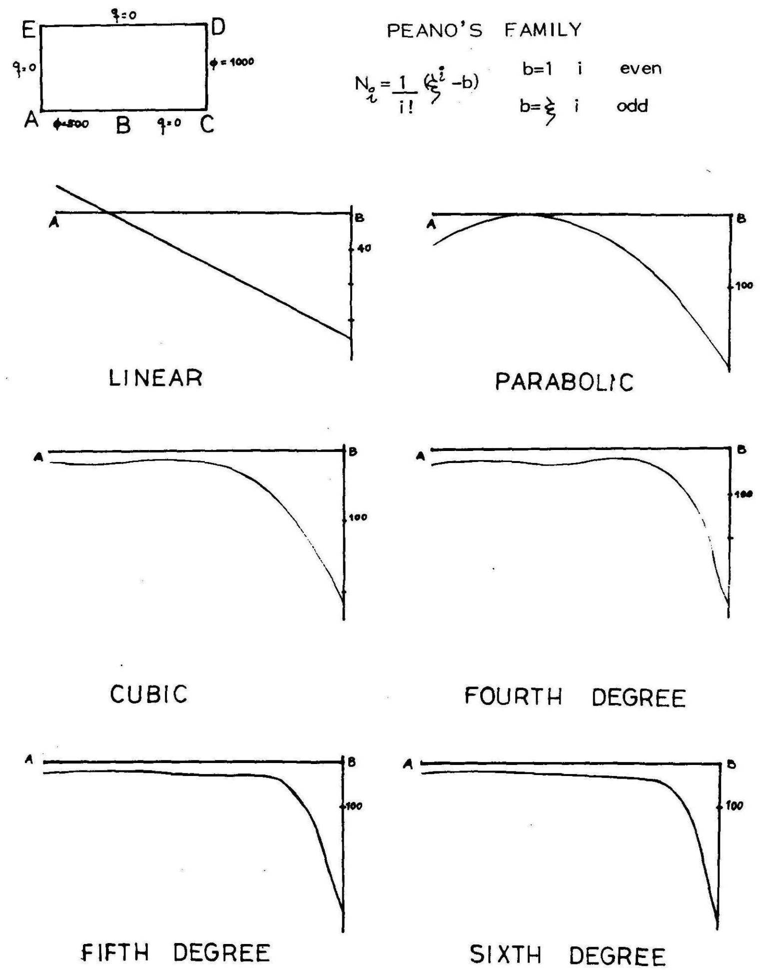
E
D
q=0
q=0
φ=1000
A
φ=500
B
q=0
C
PEANO'S FAMILY
$N_i = \frac{1}{i!}(\xi^i - b)$
b=1 i even
b=ξ i odd
A
B
40
LINEAR
A
B
100
PARABOLIC
A
B
100
CUBIC
A
B
100
FOURTH DEGREE
A
B
100
FIFTH DEGREE
A
B
100
SIXTH DEGREE

figure 3

$$N_p = \frac{1}{(p-1)!}\,\frac{1}{2^{p-2}}\,\frac{d^{p-2}}{d\xi^{p-2}}\left[(1-\xi^2)^{p-1}\right]$$

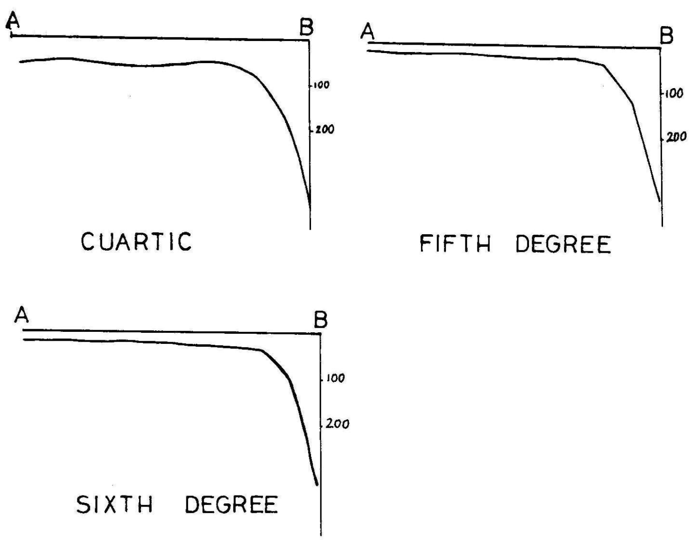

figure 4

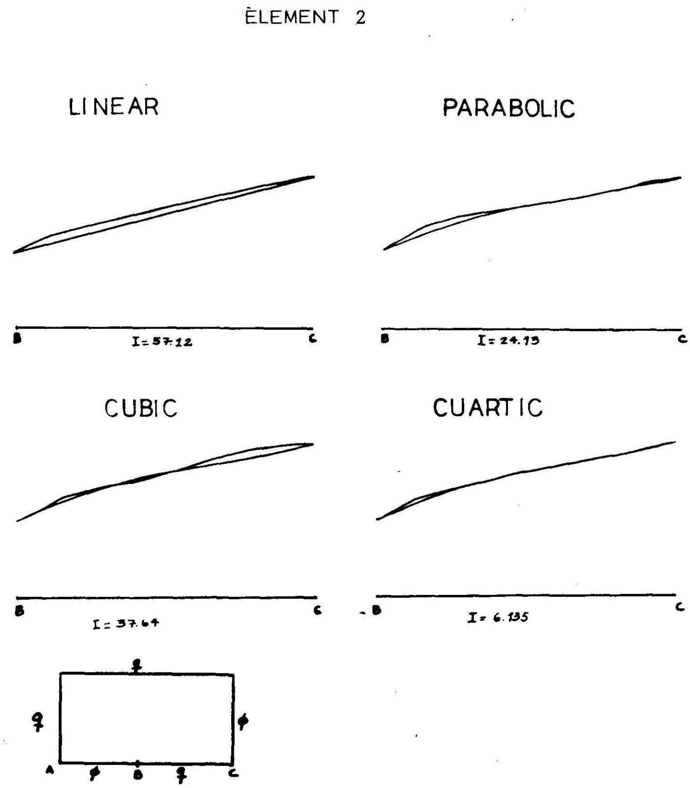
ÈLEMENT 2
LINEAR
B
I=57.12
C
PARABOLIC
B
I=24.13
C
CUBIC
B
C
I=37.64
CUARTIC
B
I=6.135
C
A
B
C

Fig. 5